INCIDENCE RELATIONS IN MULTICOHERENT SPACES. I

BY

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1. **Introduction.** In a previous paper ([1]) a study was made of the relations between the numbers of components of unions and intersections of sets and their frontiers, in a space S which was assumed to be connected, locally connected, and *unicoherent*. Here we shall investigate what happens when this last assumption is dropped. The results are naturally expressed in terms of $r(S)$, the "degree of multicoherence" of S([2]). Roughly speaking, we shall find that most of the properties in [U] can indeed be carried over, *except* those on which the previous methods were based. Thus we shall require new methods, depending on devices of "decomposition" and "approximation" (cf. §§3, 6). A consequence of this is that, whereas no separation axioms were needed in the unicoherent case treated in [U], we shall now have to assume that S is completely normal([3]). On the other hand, this extra assumption will permit some relaxing of hypotheses elsewhere (for example, slightly weaker hypotheses on frontiers of sets). As in [U], we shall be able to consider subsets of S which need not be open or closed, but are quite general.

In the first sections which follow (§§2–5) we consider the "natural" generalisations of several simple and useful point-set properties characteristic of unicoherence, notably the "Phragmén-Brouwer theorem" (cf. Kuratowski [2]). It is first shown that the expected equivalences break down in the multicoherent case, in general (§2), though some implications survive. After developing the "decomposition" technique (§3), we obtain a more complicated but valid generalisation of the Phragmén-Brouwer theorem (§4), and deduce (§5) that the "natural" generalisations, considered in §2, are after all valid if S has no local cut points.

In the later sections, after developing the method of "approximation" (§6), we investigate the properties of connected sets with disjoint frontiers, and related problems (§§7, 8). The contents of roughly the first half of [U] will

([1]) Stone [1]; this paper will be referred to as [U]. Numbers in brackets refer to the bibliography at the end of the paper.

([2]) For the fundamental properties of $r(S)$, see Eilenberg [1], [2], [3] and Whyburn [1]. The definition of $r(S)$ is repeated below.

([3]) That is, (i) S is a T_1 space, (ii) every subspace of S is normal. This could be weakened to assuming only that S is normal (and T_1), without making any essential change, but at the cost of some slight complications; cf. 6.6(3) below.

then have been generalised, and we shall leave the rest (which involves further complications) for a later paper.

Notations. Throughout, S will denote a *non-empty, connected, locally connected, completely normal*[3] space, and capital letters will usually denote subsets of S. The "degree of multicoherence" of S is defined by[4]

(1) $r(S) = \sup \{ b_0(A \cap B) \,|\, A, B$ are closed, connected, and $A \cup B = S \}$

where

(2) $b_0(X) = $ (number of components of X)—1 if this number is finite, and $b_0(X) = \infty$ otherwise.

(Thus $r(S) = 0$ if and only if S is unicoherent.) It is not necessary to assume that $r(S) < \infty$, though little would be lost by doing so. It is known (Eilenberg [1]) that $r(S)$ is finite whenever S is, for example, a polytope.

For other notations, see [U].

2. Some fundamental implications.

2.1. In [U, Theorem 1, 3.2], a number of properties of S were listed, each of which was equivalent to the unicoherence of S. We begin by determining the implications between the "obvious" generalisations of them (corresponding properties being numbered in the same way as in [U]); as was remarked above, the generalised properties are by no means equivalent in general, though they will be so in an important special case (Theorem 6, 5.1). As in the unicoherent case, several of the implications could be greatly generalised (cf. Hall-Wallace [1, Theorem 1]).

2.2. THEOREM 1. *Each of the following propositions implies the next, and* (iii) *and* (ii) *are equivalent, and* (ii)′, (iii)′, (iv), (v) *and* (vi) *are mutually equivalent; but there are no other implications between them, in general.*

(i) $r(S) \leqq n$[5].

(iii) *If C and D are disjoint connected sets such that* $\mathrm{Fr}(C) \subset \mathrm{Fr}(D)$, *then* $b_0\{\mathrm{Fr}(C)\} \leqq n$.

(ii) *If E is simple*[6], $b_0\{\mathrm{Fr}(E)\} \leqq n$.

(ii)′ *If E is open (or, equivalently, closed) and simple,* $b_0\{\mathrm{Fr}(E)\} \leqq n$.

(iii)′ *If C and D are disjoint open connected sets such that* $\mathrm{Fr}(C) = \mathrm{Fr}(D)$, *then* $b_0\{\mathrm{Fr}(C)\} \leqq n$.

(iv) *If $G_1, G_2, \cdots, G_{n+2}$ are $n+2$ open connected sets such that* $\cap \mathrm{Fr}(G_j) = 0$ *whenever $i \neq j$, and if p, q belong to the same component of* $\cap \{G_j \,|\, j \neq i\}$ *for each i, then p and q belong to the same component of* $\cap G_i$ $(i, j = 1, 2, \cdots, n+2)$.

(v) *If $F_1, F_2, \cdots, F_{n+2}$ are $n+2$ closed pairwise disjoint sets, the union of no $n+1$ of which separates two points p, q, then* $\cup F_i$ *does not separate p and q* $(1 \leqq i \leqq n+2)$.

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[4] Following Eilenberg [1]. It will be shown later (Theorem 10, 7.7) that the sets A, B in the definition of $r(S)$ could be taken to be open instead of closed.

[5] n always denotes a non-negative (finite) integer.

[6] That is, both E and $\mathrm{Co}(E)$ are connected ([U, 3.1]).

(vi) *If a (closed) set A separates two points p, q, then so does the union of some $n+1$ (or fewer) components of A.*

The proof that each proposition implies the next is a straightforward generalisation of the argument in [U, 3.2].

To prove that (ii)→(iii), let C and D be as in (iii), and let E be the component of $\mathrm{Co}(C)$ which contains D; E is simple (cf. [U, 3.1(1)]), and it is easy to see that $\mathrm{Fr}(E)=\mathrm{Fr}(C)$, so that $b_0(\mathrm{Fr}(C))\leqq n$ from (ii).

Incidentally, a similar argument shows that (iii)′ is not affected if, instead of requiring $\mathrm{Fr}(C)=\mathrm{Fr}(D)$ in it, we require only $\mathrm{Fr}(C)\subset\mathrm{Fr}(D)$.

To prove that (vi)→(ii)′, let E be an open simple set, and suppose $b_0(\mathrm{Fr}(E))>n$; thus we can write $\mathrm{Fr}(E)=F_1\cup F_2\cup\cdots\cup F_{n+2}$, where the sets F_i are closed, pairwise disjoint, and non-empty. Choose points $x_i\in F_i$, open sets $U_i\supset F_i$ such that $\overline{U}_i\cap\overline{U}_j=0$ whenever $i\neq j$, and connected neighbourhoods V_i of the respective points x_i such that $V_i\subset U_i$. There will now exist points $p\in E-\cup\overline{U}_i$, $q\in\mathrm{Co}(E)-\cup\overline{U}_i$, $y_i\in V_j\cap E$; and, since E is open and connected, there will exist closed connected sets $K_i\subset E$ containing both p and y_i[7]. Let W_i $(1\leqq i\leqq n+2)$ be an open set containing F_i such that $\overline{W}_i\subset U_i-\cup K_j$. Then $\mathrm{Fr}\{\mathrm{Co}(E)\cup\cup W_i\}$ separates p and q, and can be written as $\cup\{E\cap\mathrm{Fr}(W_i)\}$, a union of $n+2$ pairwise disjoint closed sets. By (vi), the union of some $n+1$ of them, say $\cup\{E\cap\mathrm{Fr}(W_i)\mid 2\leqq i\leqq n+2\}$, must also separate p and q. But it is disjoint from the connected set $K_1\cup V_1$ $\cup\mathrm{Co}(E)$, which contains both p and q, giving a contradiction.

2.3. To see that (ii) does *not* imply (i), take S to be a linear graph homeomorphic to a figure 8; it is easy to see that $r(S)=2$, while for every simple subset E of S we have $b_0(\mathrm{Fr}(E))\leqq 1$.

Finally, to see that (ii)′ does *not* imply (ii), we define S as follows. Let A be a 2-cell (a circle plus its interior), and take six points p_1, q_1, p_2, q_2, p_3, q_3 in cyclic order on the boundary of A. Join p_i to q_i $(i=1, 2, 3)$ by two simple arcs L_i, M_i, in such a way that all six arcs are disjoint from each other and from A except for their end points. Write $S=A\cup\cup(L_i\cup M_i)$. It is not hard to see that, if E is an *open* simple subset of S, $\mathrm{Fr}(E)$ has at most 6 components. On the other hand, A can be expressed as the union of two disjoint connected sets X, Y, in such a way that $p_i\in X$ and $q_i\in Y$ $(i=1, 2, 3)$[8]. Let H_i, K_i be proper sub-arcs of L_i, M_i respectively, containing p_i. Then, defining $E_0=X\cup\cup(H_i\cup K_i)$, we have that E_0 is simple; and $\mathrm{Fr}(E_0)$, consisting of A plus a point on each arc, has 7 components. The proof of Theorem 1 is thus complete.

3. Decompositions of sets and their frontiers.

3.1. Before continuing with the study of the properties just considered, it is convenient to derive some theorems dealing (roughly speaking) with the

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[7] This follows easily from, for example, Whyburn [1, p. 30].

[8] For example, by taking X and Y to be complementary totally imperfect subsets of E; cf. Kuratowski [1, p. 267].

problem of extending a given decomposition of the frontier of a set, to one of the whole set. As these are of some independent interest, we shall derive them in rather stronger a form than will actually be needed below.

3.2. LEMMA. *Let C be any subset of S; let $\{F_1, F_2, \cdots, F_n\}$ be any finite closed covering of $\mathrm{Fr}(C)$*[9], *and let V_j be an open set containing F_j $(1 \leqq j < n)$. Then, if N_j is the union of those components of $V_j \cap (\overline{C} \cup F_i)$ which meet F_j, the set $\bigcup \{N_j \mid j < n\} - F_n$ is open relative to $\overline{C}$.*

Let $x \in \bigcup N_j - F_n$ (where $j < n$) be given; we must construct a neighbourhood U of x such that $U \cap \overline{C} \subset \bigcup N_j - F_n$. If $x \in \mathrm{Int}(C)$, let $x \in N_j$ say, and take U to be a connected neighbourhood of x contained in $(V_j \cap C) - F_n$; clearly $U \subset N_j - F_n$. Thus we may assume $x \in \mathrm{Fr}(C)$. Let x belong to exactly k of the sets F_i; thus we may suppose $x \in F_1 \cap F_2 \cap \cdots \cap F_k - (F_{k+1} \cup \cdots \cup F_n)$. Let U be a connected neighbourhood of x contained in $V_1 \cap V_2 \cap \cdots \cap V_k$ and disjoint from $F_{k+1} \cup \cdots \cup F_n$. Then $U \cap \overline{C} = \{U \cap \mathrm{Fr}(C)\} \cup \{U \cap \mathrm{Int}(C)\}$, and so

$$
(1) \quad U \cap \overline{C} \subset F_1 \cup \cdots \cup F_k \cup \bigcup \{P_\lambda \mid P_\lambda \text{ is a component of } \mathrm{Int}\,(C) \cap V_1 \cap \cdots \cap V_k \text{ which meets } U\}.
$$

Since U also meets $\mathrm{Co}(P_\lambda)$ (in x), there exists for each λ a point $y_\lambda \in U \cap \mathrm{Fr}(P_\lambda) \subset U \cap \mathrm{Fr}(C)$, since U does not meet $\mathrm{Fr}(V_i)$ if $i \leqq k$. Thus $y_\lambda \in F_j$ for some $j \leqq k$. If Q_λ is the component of $(\overline{C} \cup F_i) \cap V_j$ which contains y_λ, clearly $P_\lambda \subset Q_\lambda \subset N_j$. Since also $F_j \subset N_j$, (1) shows that $U \cap \overline{C} \subset \bigcup N_j$ $(j < n)$; and since also $U \subset \mathrm{Co}(F_n)$, the result follows.

COROLLARY. *If V is open and $V \cap \mathrm{Fr}(C)$ is closed, the union N of those components of $V \cap \overline{C}$ which meet $\mathrm{Fr}(C)$ is open relative to $\overline{C}$.*

3.3. *Relative connectedness.* For the convenience of the reader, we repeat the definition ([U, 2.2]): A set A is *connected relative to B* if, whenever X is a non-empty subset of A which is both open and closed relative to A, X and B are not (mutually) separated. We shall make frequent use of the following obvious properties:

(1) If A is connected relative to B, and $A \supset B$, and if B is connected, then so is A.

(2) If $A \supset B \supset C$, and if A is connected relative to B, and B is connected relative to C, then A is connected relative to C.

3.4. THEOREM 2[10]. *Let C be any subset of S, and let $\{F_1, F_2, \cdots, F_n\}$ be a finite closed covering of $\mathrm{Fr}(C)$ of order at most m*[11], *such that $\bigcup F_i \neq 0$.*

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[9] That is, each F_i is closed and $\bigcup F_i \supset \mathrm{Fr}(C)$.

[10] In the case $C = S$, $m = 1$, $n = 2$, F_i connected, this theorem reduces to the characterisation of local connectedness given in Wilson [1].

[11] That is, no $m+1$ of the sets F_i have a common point.

Let open sets $U_i \supset F_i$ be given. Then there exist closed sets H_i $(1 \leqq i \leqq n)$ forming a covering of C of order at most $m+1$, such that

    (a) $H_i \supset F_i$, *and H_i is connected relative to F_i,*

    (b) $\bigcup H_i = \overline{C} \cup \bigcup F_i$,

    (c) $H_i \cap \bigcup F_k \subset U_i$,

    (d) $H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_{m+1}} \subset \mathrm{Int}(C) - \bigcup F_k$,

    (e) $\mathrm{Fr}(H_{i_1}) \cap \mathrm{Fr}(H_{i_2}) \cap \cdots \cap \mathrm{Fr}(H_{i_{m+1}}) = 0$.

$(1 \leqq i_1 < i_2 < \cdots < i_{m+1} \leqq n; \ 1 \leqq i, \ k \leqq n.)$

The proof goes by induction over n, m remaining fixed throughout. When $n = 1$, the assertion is trivial (take $H_1 = \overline{C} \cup F_1$; note that in any case $m \geqq 1$). Now assume the theorem true for $n-1$, where $n \geqq 2$.

There is evidently no loss in assuming that $F_1 \neq 0$ and that

$$(1) \quad \overline{U}_{i_1} \cap \overline{U}_{i_2} \cap \cdots \cap \overline{U}_{i_{m+1}} = 0 \quad \text{whenever } 1 \leqq i_1 < i_2 < \cdots < i_{m+1} \leqq n.$$

Choose open sets V_i such that $F_i \subset V_i \subset \overline{V}_i \subset U_i$, and let N_i be the union of those components of $V_i \cap (\overline{C} \cup \bigcup F_k)$ $(1 \leqq k \leqq n)$ which meet F_i; thus $U_i \supset \overline{N}_i \supset N_i \supset F_i$, and N_i is connected relative to F_i.

Consider the components Y_λ (say) of $C - \{\bigcup N_j \cup F_n\}$. (We shall reserve the symbol j for a suffix running from 1 to $n-1$.) Let L be the union of those sets Y_λ for which $\mathrm{Fr}(Y_\lambda)$ meets F_n, and let M be the union of the others; thus $L \cup M = C - \{\bigcup N_j \cup F_n\}$ and $L \cap M = 0$. Further, since in any case $\mathrm{Fr}(Y_\lambda) \subset \mathrm{Fr}(C) \cup \bigcup \mathrm{Fr}(N_j) \cup F_n$, we have $\mathrm{Fr}(M) \subset \bigcup \overline{N}_j$. The sets $\overline{N}_j$ form a covering of $\mathrm{Fr}(M)$ of order at most m (from (1)); hence we can apply the induction hypothesis to obtain closed sets H_j $(1 \leqq j < n)$, no $m+2$ of which have a common point, such that

    (2) $H_j \supset \overline{N}_j$; H_j is connected relative to $\overline{N}_j$; $\bigcup H_j = \overline{M} \cup \bigcup \overline{N}_j$; $H_j \cap (\overline{N}_1 \cup \cdots \cup \overline{N}_{n-1}) \subset U_j$; and $H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_{m+1}} \subset \mathrm{Int}(M)$ whenever $j_1, \cdots, j_{m+1}$ are all distinct (and less than n).

Define $H_n = \overline{L} \cup F_n$. We shall first show that $H_1, H_2, \cdots, H_n$ have the desired properties except for (e), and shall then show that property (e) can also be obtained by modifying the sets H_i.

To show that no $m+2$ of the sets H_i can have a common point, it will suffice to verify that $H_n \cap H_1 \cap H_2 \cap \cdots \cap H_{m+1} = 0$. But $H_1 \cap H_2 \cap \cdots \cap H_{m+1} \subset \mathrm{Int}(M)$, from (2); and $H_n \cap \mathrm{Int}(M) = (\overline{L} \cup F_n) \cap \mathrm{Int}(M) = 0$ since $L \cap M = 0 = F_n \cap M$.

Properties (a) and (b) are immediate, in view of 3.3(2). Thus, since (d) is an immediate consequence of (c) and (1), we have only to verify (c). Suppose first that $i < n$. Then $H_i \cap F_n \subset (\overline{M} \cap F_n) \cup \bigcup \overline{N}_j \subset \mathrm{Fr}(M) \cup \bigcup \overline{N}_j = \bigcup \overline{N}_j$; thus, from (2), $H_i \cap \bigcup F_k \subset H_i \cap \bigcup \overline{N}_j \subset U_i$. And, if $i = n$, we have $H_n \cap \bigcup F_k \subset F_n \cup \{\overline{L} \cap (\bigcup N_j - F_n)\}$; but since $\bigcup N_j - F_n$ is open relative to $\overline{C}$, from the Lemma (3.2), and is disjoint from L, it is also disjoint from $\overline{L}$, giving $H_n \cap \bigcup F_k \subset F_n \subset U_n$.

Finally, we must modify the sets H_i so that they satisfy (e). There exist

open sets $X_i \supset H_i$ such that (i) no $m+2$ of the sets $\overline{X}_i$ have a common point, (ii) $\overline{X}_i \subset U_i \cup \mathrm{Co}(\cup F_k)$. Let K_i be the union of those components of $X_i \cap (\overline{C} \cup \cup F_k)$ which meet H_i, and write $H_i^* = \overline{K}_i$. The sets H_i^* have all the properties of the sets H_i (that is, no $m+2$ of them have a common point, and (a)-(d) hold for them). Further, they satisfy (e). For suppose there exists a point $x \in \mathrm{Fr}(H_1^*) \cap \mathrm{Fr}(H_2^*) \cap \cdots \cap \mathrm{Fr}(H_{m+1}^*)$ (say); then $x \in \overline{X}_1 \cap \cdots \cap \overline{X}_{m+1}$, so from (i) $x \notin H_{m+2} \cup \cdots \cup H_n$, and from (ii) $x \in \mathrm{Int}(C)$. Thus x belongs to some H_i for $i \leq m+1$, say to H_1, and consequently belongs to some component P of $X_1 \cap \mathrm{Int}(C)$. Clearly P is open and is contained in K_1; hence $x \in \mathrm{Int}(H_1^*)$—a contradiction.

3.5. THEOREM 3. *Let C be any proper non-empty subset of S, and let X be any closed set containing $\mathrm{Fr}(C)$ and connected relative to $\mathrm{Fr}(C)$ (for example, $X = \mathrm{Fr}(C)$). Suppose $b_0(X) \geq n \geq 0$. Then there exist two closed sets H, K, each connected relative to $\mathrm{Fr}(C)$, such that $H \cup K = \overline{C} \cup X$, $\mathrm{Fr}(H) \cap \mathrm{Fr}(K) = 0$, $H \cap K \subset \mathrm{Int}(C) - X$, and $b_0(H \cap K) + b_0(\overline{C} \cup X) \geq n - 1$.*

Proof. We may write $X = F_1 \cup F_2 \cup \cdots \cup F_{n+1}$, where the sets F_i are closed, non-empty, and pairwise disjoint. By Theorem 2, there exist closed sets H_i ($1 \leq i \leq n+1$) such that

(1) $H_i \supset F_i$; H_i is connected relative to F_i; $\cup H_i = \overline{C} \cup X$; $\mathrm{Fr}(H_i) \cap \mathrm{Fr}(H_j) = 0$ and $H_i \cap H_j \subset \mathrm{Int}(C) - X$ if $i \neq j$; and $H_i \cap H_j \cap H_k = 0$ if i, j, k are all distinct.

Form a linear graph G having $n+1$ vertices h_i ($1 \leq i \leq n+1$), h_i and h_j being joined (by a 1-cell of G) if and only if $H_i \cap H_j \neq 0$. (That is, G is the nerve of the system $\{H_i\}$.) Let G have ν components G_λ ($1 \leq \lambda \leq \nu$); we may suppose that G_λ has vertices $h_{i_{\lambda-1}+1}, h_{i_{\lambda-1}+2}, \cdots, h_{i_\lambda}$ (where $0 = i_0 < i_1 < \cdots < i_\nu = n+1$). Let $E_\lambda = \cup \{H_i \mid i_{\lambda-1} < i \leq i_\lambda\}$; thus the sets $E_1, \cdots, E_\nu$ are closed, non-empty, and pairwise disjoint, and their union is $\overline{C} \cup X$, so that

(2) $b_0(\overline{C} \cup X) \geq \nu - 1$.

Now let T_λ be a connected subgraph of G_λ having the same vertices as G_λ and as few 1-cells as possible; thus T_λ is a tree having $i_\lambda - i_{\lambda-1}$ vertices, and consequently having $(i_\lambda - i_{\lambda-1} - 1)$ 1-cells. There is a unique path joining each vertex h_i of G_λ (other than $h_{i_{\lambda-1}+1}$) to $h_{i_{\lambda-1}+1}$ in T_λ; let $\mathcal{K}_\lambda$ be the set of all vertices of G_λ for which this path has an odd number of 1-cells, and let $\mathcal{H}_\lambda$ consist of the remainder. We may suppose the notation so chosen that $h_i \in \mathcal{H}_\lambda$ for $i_{\lambda-1} < i \leq j_\lambda$, and $h_i \in \mathcal{K}_\lambda$ for $j_\lambda < i \leq i_\lambda$ (where $i_{\lambda-1} < j_\lambda \leq i_\lambda$; naturally $\mathcal{K}_\lambda$ may be empty). Define $H = \cup \{H_p \mid i_{\lambda-1} < p \leq j_\lambda, 1 \leq \lambda \leq \nu\}$, $K = \cup \{H_q \mid j_\lambda < q \leq i_\lambda, 1 \leq \lambda \leq \nu\}$. From (1), H and K are closed sets, connected relative to $\mathrm{Fr}(C)$, and $\mathrm{Fr}(H) \cap \mathrm{Fr}(K) = 0$, $H \cup K = \overline{C} \cup X$, $H \cap K \subset \mathrm{Int}(C) - X$. Further, $H \cap K = \cup \{H_p \cap H_q\}$, a union of closed sets which are pairwise disjoint (for different *pairs* (p, q)), from (1). And the number of these sets $H_p \cap H_q$ which are non-empty is the total number of 1-cells of all the G_λ's which have one

end point in $\mathcal{H}_\lambda$ and the other in K_λ. Since every 1-cell in T_λ is included here, by construction, we obtain

$$b_0(H \cap K) + 1 \geqq \sum (i_\lambda - i_{\lambda-1} - 1) = n + 1 - \nu;$$

and the result now follows from (2).

3.6. THEOREM 4. *Let C be any subset of S, having components $\{D_\mu\}$, and X be any closed set containing* $\mathrm{Fr}(C)$ *and connected relative to* $\mathrm{Cl}(\mathrm{Co}(C))$[12]. *Then*

$$b_0(X) \leqq \sum b_0\{\mathrm{Fr}(D_\lambda)\} + b_0(\overline{C} \cup X)[13].$$

Let $X = Y_1 \cup Y_2 \cup \cdots \cup Y_{k+1}$, where the sets Y_i are closed, non-empty, and pairwise disjoint; it will evidently suffice to prove that

(1) $$k \leqq \sum b_0\{\mathrm{Fr}(D_\lambda)\} + b_0(\overline{C} \cup X),$$

and in doing this we can certainly assume the right-hand side to be finite, so that C can have only a finite number of components, say $D_1, D_2, \cdots, D_n$ whose frontiers are *not* connected. Let $\{E_\mu\}$ be the remaining components of C (with connected frontiers). The proof will go by induction over n.

Suppose first $n = 0$. Write $H_i = Y_i \cup \bigcup \{E_\mu | \mathrm{Fr}(E_\mu) \subset Y_i\}$ $(1 \leqq i \leqq k+1)$. Clearly $\bigcup H_i = X \cup C = X \cup \overline{C}$, and each H_i is closed and non-empty. Further, the sets H_i are pairwise disjoint, since if (say) $H_1 \cap H_2 \neq 0$, there is no loss of generality in assuming that Y_2 meets some E_μ with $\mathrm{Fr}(E_\mu) \subset Y_1$, and then $Y_2 \cap \mathrm{Int}(E_\mu)$ is a non-empty and relatively open-closed subset of X which is separated from $\mathrm{Cl}(\mathrm{Co}(C))$, contrary to hypothesis. Hence $b_0(\overline{C} \cup X) \geqq k$, establishing (1) in this case.

Now suppose that (1) holds for $n - 1$ (for all sets C and X satisfying the conditions), and write $C' = C - D_n$. Then X is a fortiori connected relative to $\mathrm{Cl}(\mathrm{Co}(C'))$, and contains $\mathrm{Fr}(C')$, so that the hypothesis of induction gives the existence of a (finite) non-negative integer $r \leqq b_0(\overline{C}' \cup X)$ such that

(2) $$k \leqq r + \sum b_0\{\mathrm{Fr}(D_i)\} \qquad (1 \leqq i \leqq n-1).$$

We can write $C' \cup X = \overline{C}' \cup X = K_1 \cup K_2 \cup \cdots \cup K_{r+1}$, a union of $r+1$ closed, non-empty, pairwise disjoint sets. Let $\mathrm{Fr}(D_n)$ meet exactly s of the sets K_j, say $K_1, \cdots, K_s$ (where $0 \leqq s \leqq r+1$). Then, since $\mathrm{Fr}(D_n) \subset X \subset \bigcup K_j$, we have $b_0\{\mathrm{Fr}(D_n)\} \geqq s - 1$.

Again, $\overline{D}_n$ is disjoint from K_j if $j > s$; for otherwise $K_j \cap \mathrm{Int}(D_n)$ is a nonempty and relatively open-closed subset of X which is separated from $\mathrm{Cl}(\mathrm{Co}(C))$, contrary to hypothesis. Accordingly, we have $\overline{C} \cup X = (\overline{D}_n \cup K_1 \cup K_2 \cup \cdots \cup K_s) \cup K_{s+1} \cup \cdots \cup K_{r+1}$, a union of $r - s + 2$ pairwise disjoint, closed, non-empty sets, showing that $b_0(\overline{C} \cup X) \geqq r - s + 1$.

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[12] To simplify the printing of complicated expressions, we shall sometimes write $\mathrm{Cl}(\dot{x})$ instead of $\bar{x}$.

[13] If S is unicoherent, it can be shown that this inequality becomes an equality.

Thus $b_0(\overline{C}\cup X)+b_0\{\mathrm{Fr}(D_n)\}\geqq r$, whence (1) follows from (2) in the present case also. This establishes (1) for all n, proving the theorem.

4. Extension of the Phragmén-Brouwer Theorem.

4.1. Let A be an arbitrary connected set, and let C be any component of $\mathrm{Co}(A)$ (so that C is simple). We have seen (Theorem 1, 2.2) that $\{r(S)\leqq n\}$ $\rightarrow\{b_0(\mathrm{Fr}(C))\leqq n$, for all A and $C\}$, though the converse implication is false in general([14]). But a stronger result, which does characterize $r(S)$, can be obtained by considering *all* the components of $\mathrm{Co}(A)$:

THEOREM 5. *If A is any subset of S such that $\overline{A}$ is connected and if $\{C_\lambda\}$ are the components of $\mathrm{Co}(A)$, then*

(1) $$\sum b_0\{\mathrm{Fr}(C_\lambda)\} \leqq r(S).$$

Conversely, if for some fixed n we have

(2) $$\sum b_0\{\mathrm{Fr}(C_\lambda)\} \leqq n$$

whenever $\{C_\lambda\}$ are the components of the complement of an open connected set A which is such that $b_0(\mathrm{Co}(A)\} <\infty$, then $r(S)\leqq n$. In this statement, the word "open" may be replaced by "closed."

4.2. In proving (1), we can assume $\mathrm{Fr}(C_\lambda)\neq 0$—for otherwise $C_\lambda=S$ and (1) holds trivially—and can thus take for each λ an integer n_λ such that $0\leqq n_\lambda\leqq b_0\{\mathrm{Fr}(C_\lambda)\}$. It will clearly suffice to prove

(3) $$\sum n_\lambda \leqq r(S).$$

From Theorem 3 (3.5) there exist closed sets H_λ, K_λ, connected relative to $\mathrm{Fr}(C_\lambda)$, such that $H_\lambda\cup K_\lambda=\overline{C}_\lambda$, $H_\lambda\cap K_\lambda\subset\mathrm{Int}\,(C_\lambda)$, and $b_0(H_\lambda\cap K_\lambda)\geqq n_\lambda-1$. Write $L=\overline{A}\cup\mathrm{U}H_\lambda$, $M=\overline{A}\cup\mathrm{U}K_\lambda$. Then L is connected relative to $\overline{A}$, and is therefore connected; similarly, so is M. Again, *L is closed.* For suppose $x\in\overline{L}-L$; then $x\notin\overline{A}$, so $x\in\mathrm{Int}(C_\lambda)$ for some λ, whence x belongs to the closure of $\mathrm{Int}(C_\lambda)\cap L\subset H_\lambda$, and so $x\in H_\lambda\subset L$, a contradiction. Similarly M is closed. Since $L\cup M=S$, we have $b_0(L\cap M)\leqq r(S)$.

Now $L\cap M=\overline{A}\cup\mathrm{U}(H_\lambda\cap K_\mu)=\overline{A}\cup\mathrm{U}(H_\lambda\cap K_\lambda)$, since if $\lambda\neq\mu$ we have $H_\lambda\cap K_\mu\subset\overline{C}_\lambda\cap\overline{C}_\mu\subset\overline{A}$. Further, $H_\lambda\cap K_\lambda=\mathrm{Int}(C_\lambda)\cap(L\cap M)$, so that $H_\lambda\cap K_\lambda$ is both open and closed relative to $L\cap M$. Thus $L\cap M$ has at least $1+\sum n_\lambda$ components (consisting of $\overline{A}$ together with the separate components of the sets $H_\lambda\cap K_\lambda$); that is, $r(S)\geqq b_0(L\cap M)\geqq \sum n_\lambda$, Q.E.D.

COROLLARY. *If $\overline{A}$ is connected and $r(S)<\infty$, all but at most a finite number of the components of $\mathrm{Co}(A)$ have connected frontiers.*

4.3. We shall derive the remainder of Theorem 5 from the following sharper result:

([14]) It holds if $n=0$, the resulting equivalence constituting the Phragmén-Brouwer theorem; cf. Kuratowski [2] and [U, Theorem 1, 3.2].

THEOREM 5a. *If $r(S)>n$, there exists a connected set A such that $b_0\{\mathrm{Co}(A)\}$ $<\infty$ and $b_0\{\mathrm{Fr}(A)\}>b_0\{\mathrm{Co}(A)\}+n$; further, A can be taken to be open, or to be closed.*

In fact, once Theorem 5a is established, Theorem 4 (3.6) shows that (2) above will be violated, and the proof of Theorem 5 will be complete.

First we construct an *open* set A fulfilling the requirements of Theorem 5a. Since $r(S)>n$, there exist closed connected sets A_1, B_1 such that $A_1\cup B_1 = S$ and $A_1\cap B_1 = D_1\cup D_2\cup\cdots\cup D_{n+2}$, where the sets D_i are closed, non-empty, and pairwise disjoint. There exist open sets $U_i\supset D_i$ such that $\overline{U}_i\cap\overline{U}_j = 0$ if $i\neq j$; and we can further suppose that every component of U_i meets D_i (by discarding the other components of U_i). Write $A'=A_1\cup\mathsf{U}U_i$, $B'=B_1\cup\mathsf{U}U_i$, and $X=\mathsf{U}\overline{U}_i$. Clearly A' and B' are connected open sets, and $\mathrm{Fr}(A')\subset X$. Further, if V is any component of any U_i, then since $V\cap B'$ and $B'-V$ are non-empty, they are not separated, from which it readily follows that $\mathrm{Fr}(V)$ meets $\mathrm{Fr}(A')$. Hence it is easy to see that X is connected relative to $\mathrm{Fr}(A')$. Thus we may apply Theorem 4 (3.6) to the set $\mathrm{Co}(A')$ with components $\{C_\lambda\}$ (say), obtaining

$$b_0(X) \leqq \sum b_0\{\mathrm{Fr}(C_\lambda)\} + b_0\{\mathrm{Co}(A')\cup X\}.$$

But $\mathrm{Co}(A')\cup X=\overline{B}'$, and is therefore connected. Accordingly, since $b_0(X)>n$, there exist a finite number of components C_λ of $\mathrm{Co}(A')$, say $C_1, C_2, \cdots, C_k$, such that $\sum b_0\{\mathrm{Fr}(C_i)\}>n$ $(1\leqq i\leqq k)$. Let $A=\mathrm{Co}(C_1\cup C_2 \cup\cdots\cup C_k)$. It is easy to see that A is connected and open, that the components of $\mathrm{Co}(A)$ are simply the closed sets C_i $(1\leqq i\leqq k)$, and that the components of $\mathrm{Fr}(A)$ are precisely the components of the sets $\mathrm{Fr}(C_i)$. Thus we obtain $b_0\{\mathrm{Fr}(A)\} = \sum\{b_0(\mathrm{Fr}(C_i))+1\}-1>n+k-1$, where $k-1 = b_0(\mathrm{Co}(A))$, as desired.

4.4. From the open set A just constructed, we now derive a *closed* set A^* having the same properties. We may assume $b_0\{\mathrm{Fr}(C_i)\}<\infty$ $(1\leqq i\leqq k)$, since otherwise we merely take $A^*=C_i$. Thus $\mathrm{Fr}(A)$ will have a finite number of components, say $F_1, F_2, \cdots, F_m$, where $m>k+n$. Take open sets $V_i\supset F_i$ such that $\overline{V}_i\cap\overline{V}_j=0$ whenever $i\neq j$, and let N_i denote the union of those components of $V_i\cap\overline{A}$ which meet F_i; thus $F_i\subset N_i\subset V_i\cap\overline{A}$, and N_i is open relative to $\overline{A}$ (3.2, Corollary). Hence there exist points $x_i\in N_i\cap A$; and since A is open and connected there exists a closed connected set $L\subset A$ containing all the m points x_i. Write $E=L\cup\mathsf{U}\{K_\lambda\,|\,K_\lambda$ is a component of $A-\mathsf{U}N_i$ which meets $L\}$. Clearly E is connected, and it is easy to see that $\overline{E}\subset A$. Finally, define $A^*=\overline{E}\cup\mathsf{U}\{P_\mu\,|\,P_\mu$ is a component of $A-\overline{E}$ such that $\mathrm{Fr}(P_\mu)\subset\mathrm{Fr}(E)\}$. One readily verifies that A^* is a closed connected subset of A. Further, since in any case $\mathrm{Fr}(P_\mu)\subset\mathrm{Fr}(A)\cup\mathrm{Fr}(E)$, $A-A^*$ is a union of components P_ν of $A-\overline{E}$ which are such that $\mathrm{Fr}(P_\nu)$ meets $\mathrm{Fr}(A)$; thus $\mathrm{Co}(A^*)$ is connected relative to $\mathrm{Co}(A)$, and so has at most k components. Again, x_i will belong to a component W_i (say) of N_i; W_i meets A^* (in x_i)

and meets $F_i \subset \mathrm{Co}(A^*)$, and so meets $\mathrm{Fr}(A^*)$. Thus $\mathrm{Fr}(A^*)$ meets each $\overline{N}_i$; and since $\mathrm{Fr}(A^*) \subset \mathrm{Fr}(E) \subset \cup \overline{N}_i$, a union of m disjoint closed sets, $\mathrm{Fr}(A^*)$ must have at least m components. Therefore $b_0\{\mathrm{Fr}(A^*)\} > b_0\{\mathrm{Co}(A^*)\} + n$.

5. Equivalences for spaces without local cut points.

5.1. THEOREM 6. *If S has no local cut points, all the properties* (i)–(vi) *of Theorem* 1 (2.2) *are equivalent.*

It will suffice to prove that (ii)′$\rightarrow$(i); thus we may assume $r(S) > n$, and must construct a simple open subset E of S such that $b_0\{\mathrm{Fr}(E)\} > n$. To avoid obscuring the idea of the proof by "point-set" complications, we shall first carry it out assuming S to be a Peano space, and shall then briefly indicate how the argument would be modified in the general case.

5.2. *Proof if S is a Peano space.* By Theorem 5a (4.3), there exists an open connected set A such that (i) $\mathrm{Co}(A)$ has only a finite number of components $C_1, \cdots, C_k$ and (ii) $b_0\{\mathrm{Fr}(A)\} > b_0\{\mathrm{Co}(A)\} + n$. If $k = 1$, there is nothing to prove. If $k > 1$, we shall show that A may be replaced by a set B having similar properties but a smaller value of k; iteration will then lead to $k = 1$, establishing the theorem.

We may assume $b_0\{\mathrm{Fr}(C_i)\} < \infty$, since otherwise we merely take $E = \mathrm{Co}(C_i)$. Hence $\mathrm{Fr}(A)$ consists of a finite number of components $F_1, \cdots, F_m$, where (as in 4.4) $m > k + n$. Choose connected open sets $U_i \supset F_i$ such that $\overline{U}_i \cap \overline{U}_j = 0$ whenever $i \neq j$, and choose points $x_i \in U_i \cap A$. Since S has no local cut points, there exists (see Whyburn [2]) a simple closed curve J in A which contains all the m points x_i. There exists, for each i, a simple arc L_i joining x_i to F_i in U_i; and, on replacing L_i by a sub-arc if necessary, we can ensure that $L_i \cap J$ is a single point, say y_i, and that $L_i \cap F_i$ is a single point z_i. Write $L_i' = L_i - (z_i)$; thus $L_i' \subset A$. By a suitable change of numbering, we can suppose that $y_1, y_2, \cdots, y_m$ are in cyclic order on J, and that $F_1 \subset C_1$ and $F_2 \subset C_2$.

Let H, K denote the two open arcs $(y_1 y_2)$ of J, the points $y_3, \cdots, y_m$ belonging to K. Consider the sets $R = H \cup L_1 \cup L_2$, $T = K \cup L_3' \cup \cdots \cup L_m'$. Clearly R and T are connected sets, R is closed, $R \cap T = 0$, and $T \subset A$. Let B be the component of $A - R$ which contains T. Since $R \cap A$ is connected, B is simple relative to A—that is, B and $A - B$ are both connected; further, B is open; and since the frontier of B relative to A is contained in R, $\mathrm{Fr}(B)$ meets R.

We have $\mathrm{Co}(B) = \{(A - B) \cup R \cup C_1 \cup C_2\} \cup C_3 \cup \cdots \cup C_k$, in which $\{(A - B) \cup R \cup C_1 \cup C_2\}$ is connected; hence $\mathrm{Co}(B)$ has at most $k - 1$ components. And $\mathrm{Fr}(B) \subset \mathrm{Fr}(A) \cup R = \{F_1 \cup R \cup F_2\} \cup F_3 \cup \cdots \cup F_m$, a union of $m - 1$ disjoint closed sets, each of which meets $\mathrm{Fr}(B)$ (for if $i > 2$, $z_i \in \overline{T} \cap F_i \subset \overline{B} \cap F_i = \mathrm{Fr}(B) \cap F_i$). Hence $\mathrm{Fr}(B)$ has at least $m - 1$ components, so $b_0\{\mathrm{Fr}(B)\} > b_0\{\mathrm{Co}(B)\} + n$.

Thus B has all the properties which were required of A, and $\mathrm{Co}(B)$ has fewer components than $\mathrm{Co}(A)$. By repeating the argument we finally obtain a simple open set E with these properties; and then $b_0\{\mathrm{Fr}(E)\}>n$. Q.E.D.

5.3. *Extension to general S.* In the general case, the arcs used in the above construction will no longer be available; but they can be replaced by suitable closed connected sets, and the argument is then essentially the same as before. The main result to be proved, in order to replace the configuration of arcs used above, is:

(1) Let A be a connected open subset of S (having no local cut points), and let $U_1, \cdots, U_m$ be open connected sets such that $\overline{U}_i \cap \overline{U}_j = 0$ $(i \neq j)$ and each U_i meets $\mathrm{Fr}(A)$. Then there exist connected open sets V_i such that (a) $V_i \subset A$ and $\overline{V}_i \subset U_i$, (b) $\mathrm{Fr}(V_i)$ meets $\mathrm{Fr}(A)$, and (c) after a suitable renumbering of the sets V_i, there exist pairwise disjoint closed connected subsets $H_1, \cdots, H_m$ of A such that H_i meets $\overline{V}_i$ and $\overline{V}_{i+1}$, but does not meet $\overline{V}_j$ if $j \neq i, i+1$ $(i, j = 1, \cdots, m; V_{m+1} = V_1)$.

This can be proved by induction over m, the main lemma being the following sharpening of one due to Ayres and Whyburn (Ayres [1]):

(2) Let A be a connected open set having no cut point, let E_1, E_2 be subsets of A such that $A \cap \overline{E}_1 \cap \overline{E}_2 = 0$, and let p, q be distinct points of A such that neither of the sets $A \cap \overline{E}_1$, $A \cap \overline{E}_2$, disconnects([15]) the other from p in A. Then there exist two disjoint closed connected sets H, K, such that H joins one of E_1 and E_2, say E_i, to p, K joins the other, E_j, to q, $H \subset A - \overline{E}_j$, and $K \subset A - \overline{E}_i$.

The proof of (2) follows the lines of the argument in Whyburn [1, pp. 93, 94]. The detailed proofs of (1) and (2) are, however, long and tedious, and are therefore omitted.

6. Frontier properties and approximation.

6.1. In subsequent arguments we shall have to replace two open (or closed) sets A, B by approximating closed (or open) ones, and it will be important that this can be done without losing certain frontier properties which A and B may have. As we now show, this can be done; in fact, the relevant frontier properties will actually be improved by the process. Thus, as a by-product, we shall later be able to state theorems under slightly weaker hypotheses on A and B than would otherwise be the case.

DEFINITION. We shall say that two sets A, B are "*strongly approximable by open sets having property* (P)" if there exist open sets $U \supset A$ and $V \supset B$ such that every two open sets A^*, B^* such that $A \subset A^* \subset U$ and $B \subset B^* \subset V$ have property (P)([16]).

([15]) A set X "*disconnects*" sets Y, Z if every connected set which meets both Y and Z must also meet X.

([16]) Though the theorems which follow all deal with *strong* approximability, it will be clear that in the applications a much weaker notion of approximability would usually suffice. Cf. 7.8 below.

6.2. LEMMA. *If* $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cup B) = 0$, *then A and B are strongly approximable by open sets having disjoint frontiers.*

There exist open sets $L \supset \mathrm{Fr}(A)$, $M \supset \mathrm{Fr}(B)$, $N \supset \mathrm{Fr}(A \cup B)$ such that $\overline{L} \cap \overline{M} \cap \overline{N} = 0$. Write $X = (A \cup L) \cap (A \cup B \cup N)$, $Y = (B \cup M) \cap (A \cup B \cup N)$; it is easy to see that X and Y are open sets containing A and B respectively. Suppose now that A^* and B^* are open sets such that $A \subset A^* \subset X$ and $B \subset B^* \subset Y$; we must prove $\mathrm{Fr}(A^*) \cap \mathrm{Fr}(B^*) = 0$. If not, let $x \in \mathrm{Fr}(A^*) \cap \mathrm{Fr}(B^*)$; then $x \in \overline{A^*} - A^* \subset \overline{X} - A \subset \mathrm{Cl}(A \cup L) - A \subset \overline{L}$. Similarly $x \in \overline{M}$. Again, $x \in \overline{X} \subset \mathrm{Cl}(A \cup B \cup N)$; but $x \notin A$, and similarly $x \notin B$, so that $x \in \overline{N}$. This contradicts the choice of L, M, N.

6.3. THEOREM 7. *If $A - B$ and $B - A$ are separated([17]) and $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cup B) = 0$, then, given any open set $W \supset A \cap B$, A and B are strongly approximable by open sets whose frontiers are disjoint and whose intersection is contained in W.*

With the same sets L, M, N as in the lemma preceding, write $U^* = A \cup \{ N \cap (L - \overline{B}) \} \cup W$, $V^* = B \cup \{ N \cap (M - \overline{A}) \} \cup W$. It is not hard to see that U^* and V^* are open; and clearly $A \subset U^*$, $B \subset V^*$, and $U^* \cap V^* = W$. The sets $U = X \cap U^*$ and $V = Y \cap V^*$ are now the open neighbourhoods of A and B required; in fact, if A^*, B^* are open and $A \subset A^* \subset U$ and $B \subset B^* \subset V$, we have $\mathrm{Fr}(A^*) \cap \mathrm{Fr}(B^*) = 0$, from the lemma, and $A^* \cap B^* \subset U^* \cap V^* = W$.

6.4. REMARK. The conditions on A and B in Theorem 7—that $A - B$ and $B - A$ are separated and $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cup B) = 0$—are together equivalent to the single condition $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \subset A \cap B \cap \mathrm{Int}(A \cup B)$. They therefore certainly hold if $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) = 0$. Conversely, they imply $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) = 0$ if either A or B is open.

6.5. THEOREM 7a. *If $A - B$ and $B - A$ are separated, and also $A \cap B$ and $\mathrm{Co}(A \cup B)$ are separated, then, given any open set W containing $A \cap B$, A and B are strongly approximable by open sets A^*, B^*, satisfying $A^* \cap B^* \subset W$ and $\mathrm{Fr}(A^*) \cap \mathrm{Fr}(B^*) \cap \mathrm{Fr}(A^* \cap B^*) = 0$.*

There exist open sets L, M, X, Y such that $A - B \subset L$, $B - A \subset M$, $L \cap M = 0$, $A \cap B \subset X$, $\mathrm{Co}(A \cup B) \subset Y$, and $X \cap Y = 0$. (Here, for the first time, we use the *complete* normality of S.) Write $U = L \cup (X \cap W)$, $V = M \cup (X \cap W)$; clearly U, V are open sets containing A, B, respectively, and $U \cap V \subset W \cap X$. Let A^*, B^* be any open sets such that $A \subset A^* \subset U$ and $B \subset B^* \subset V$. Clearly $A^* \cap B^* \subset W$; and we have $\mathrm{Fr}_2(A^*) \cap \mathrm{Fr}(B^*) \cap \mathrm{Fr}(A^* \cap B^*) \subset \mathrm{Co}(A^*) \cap \mathrm{Co}(B^*) \cap \mathrm{Cl}(A^* \cap B^*) \subset Y \cap \overline{X} = 0$.

6.6. REMARKS.

(1) The hypotheses on A and B in Theorem 7a are implied by those in

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([17]) This condition (introduced by Wallace [1]) will hold if A and B are both closed, or both open, relative to $A \cup B$, or if $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) = 0$, and so on.

Theorem 7—an implication which becomes an equivalence if A and B are both closed. Again, if they are satisfied by A and B, they are satisfied also by $\mathrm{Co}(A)$ and $\mathrm{Co}(B)$; hence they are also implied by: $\{A - B$ and $B - A$ are separated and $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cap B) = 0\}$—an implication which becomes an equivalence if A and B are both open.

(2) The converse of Theorem 7a is also true, provided that S is metrisable (or, more generally, has a locally countable base—some such proviso is essential) and has no isolated points. The converse of Theorem 7 is false in general, but holds if A and B are closed.

(3) The arguments in the present section made no use of the connectedness or local connectedness of S; they are therefore valid (except for the converse of Theorem 7a) in an arbitrary completely normal space. Further, the only use made of *complete* normality—and the only essential use of it in the whole paper—was in proving Theorem 7a; elsewhere, normality would have sufficed. Thus our assumptions on S could be weakened throughout to requiring only that S be a non-empty, connected, locally connected, and normal (T_1) space, provided that two minor adjustments are made in what follows: (a) every appeal to Theorem 7a must be replaced by an appeal to Theorem 7, the hypotheses on A and B being strengthened accordingly, and (b) for general sets (neither open nor closed), $b_0(X)$ should be replaced by a similar function based on the "normal components" of X, which are defined in an obvious way in terms of the notion of "normal connectedness" given in [U, 4.3].

7. Connected sets with "almost disjoint" frontiers.

7.1. In [U] it was shown that the defining property of unicoherence could be extended to two connected sets which were open and covered S, or more generally which had disjoint frontiers (and were not necessarily open). Here we shall obtain a similar extension; but the process can now (in view of §6) be carried even further, applying to sets which can be approximated by sets having disjoint frontiers (Theorem 9, 7.5)—a fairly inclusive result, since it includes also, for example, the case in which the two connected sets are closed and cover S. It readily follows that the sets used to define $r(S)$ could be taken to be open, instead of closed, without affecting the definition (Theorem 10, 7.7). We conclude §7 by showing that Theorem 9 also holds under modified conditions, and deducing some implications between frontier relations for the case in which S is unicoherent.

7.2. The starting point is a generalisation of the "extension theorem" [U, Theorem 4, 4.8]:

THEOREM 8. *Let A and B be any sets, not both empty, such that* $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cup B) = 0$. *Then there exist sets A^*, B^*, such that* $A^* \supset A$, $B^* \supset B$, $A^* \cup B^* = S$, $A \cap B$ *is both open and closed relative to* $A^* \cap B^*$, *A^* and B^* are connected relative to A and B respectively, and* $\mathrm{Fr}(A^*) \cap \mathrm{Fr}(B^*)$ $\subset \mathrm{Fr}(A) \cap \mathrm{Fr}(B)$. *Moreover, A^* and B^* can be chosen to be closed if A and B*

are closed, and open if A and B are open([18]).

Proof. Write $C = \text{Co}(A \cup B)$, $F_1 = \text{Fr}(C) \cap \text{Fr}(A)$, and $F_2 = \text{Fr}(C) \cap \text{Fr}(B)$; thus $F_1 \cap F_2 = 0$ and $F_1 \cup F_2 = \text{Fr}(C)$. The case in which $F_1 \cup F_2 = 0$ is easily seen to be trivial; hence we may apply Theorem 2 (3.4), obtaining closed sets H_1, H_2, such that $H_i \supset F_i$ and H_i is connected relative to F_i ($i = 1, 2$), $H_1 \cup H_2 = \overline{C}$, $H_1 \cap H_2 \subset \text{Int}(C)$, and $\text{Fr}(H_1) \cap \text{Fr}(H_2) = 0$. Now define $A^* = A \cup H_1$, $B^* = B \cup H_2$. Clearly A^*, B^* are connected relative to A, B, respectively, and $A^* \cup B^* = S$. Further, it is easy to see that $A^* \cap B^* = (A \cap B) \cup (H_1 \cap H_2)$, where $H_1 \cap H_2 \subset \text{Ext}(A \cup B)$, so that $A \cap B$ is both open and closed relative to $A^* \cap B^*$. And we have $\text{Fr}(A^*) \cap \text{Fr}(B^*) \subset \{\text{Fr}(A) \cup \text{Fr}(H_1)\} \cap \{\text{Fr}(B) \cup \text{Fr}(H_2)\} \subset \text{Fr}(A) \cap \text{Fr}(B)$ since $\text{Fr}(H_1) \cap \text{Fr}(H_2) = 0$ and $H_1 \cap F_2 = 0 = H_2 \cap F_1$.

Obviously A^* will be closed if A is, and similarly for B^*. All that remains is to deal with the case in which A and B are both open. Here the definition of A^* and B^* must be modified. There exists an open set U, connected relative to $H_1 \cap H_2$, such that $H_1 \cap H_2 \subset U$ and $\overline{U} \subset \text{Int}(C)$. Replace A^* by $A^* \cup U$ and B^* by $B^* \cup U$; it is easy to verify that these sets are open and have all the properties required.

REMARK. It is also easy to see that the (unmodified) sets will satisfy: $A^* - B^*$ and $B^* - A^*$ are separated if $A - B$ and $B - A$ are separated.

7.3. LEMMA. *If A and B are closed connected sets such that* $\text{Fr}(A) \cap \text{Fr}(B) \cap \text{Fr}(A \cup B) = 0$, *then* $b_0(A \cap B) \leqq r(S)$.

For we have only to apply Theorem 8, giving $b_0(A \cap B) \leqq b_0(A^* \cap B^*) \leqq r(S)$.

7.4. LEMMA. *If A and B are open connected sets such that* $\text{Fr}(A) \cap \text{Fr}(B) \cap \text{Fr}(A \cap B) = 0$, *then* $b_0(A \cap B) \leqq r(S)$.

We deduce this from 7.3 by "approximation." Suppose the lemma is false; then $r(S)$ is finite, and equal to n, say, and we can pick points $p_1, p_2, \cdots, p_{n+2}$ belonging to different components of $A \cap B$. There exist closed connected sets $L \subset A$ and $M \subset B$ both of which contain all the $n+2$ points p_i. Now, the closed sets $\text{Co}(A)$ and $\text{Co}(B)$ satisfy the hypotheses of Theorem 7 (6.3), and $\text{Co}(L)$ and $\text{Co}(M)$ are open sets containing them. Hence (Theorem 7) there exist open sets X, Y, with disjoint frontiers, such that $\text{Co}(A) \subset X \subset \text{Co}(L)$ and $\text{Co}(B) \subset Y \subset \text{Co}(M)$. Let H be the component of $\text{Co}(X)$ which contains L, and similarly let K be the component of $\text{Co}(Y)$ which contains M. Then H and K are closed, and $\text{Fr}(H) \cap \text{Fr}(K) \subset \text{Fr}(X) \cap \text{Fr}(Y) = 0$. By 7.3, $b_0(H \cap K) \leqq n$. But $L \cap M \subset H \cap K \subset A \cap B$; hence some

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([18]) By analogy with [U, 4.8], one might expect that the further conditions $A^* \cap B^* = A \cap B$, $\text{Fr}(A^*) \subset \text{Fr}(A)$ and $\text{Fr}(B^*) \subset \text{Fr}(B)$ could also be imposed; but it is easy to see from examples that this is impossible, in general.

two of the points p_i must belong to the same component of $H \cap K$, and therefore to the same component of $A \cap B$—a contradiction.

7.5. THEOREM 9. *Let A and B be any connected sets such that $A - B$ and $B - A$ are separated, and also $A \cap B$ and $\mathrm{Co}(A \cup B)$ are separated. Then $b_0(A \cap B) \leqq r(S)$.*

This follows from 7.4 by "approximation" again. Let n be any non-negative integer such that $n \leqq b_0(A \cap B)$; it will suffice to prove that $n \leqq r(S)$. We can write $A \cap B = C_1 \cup C_2 \cup \cdots \cup C_{n+1}$, where the sets C_i are non-empty and every two of them are separated. Hence there exist open sets $W_i \supset C_i$ such that $W_i \cap W_j = 0$ whenever $i \neq j$. Let $W = \cup W_i$, and apply Theorem 7a (6.5); there will exist open sets $A^* \supset A$ and $B^* \supset B$ such that $A^* \cap B^* \subset W$ and $\mathrm{Fr}(A^*) \cap \mathrm{Fr}(B^*) \cap \mathrm{Fr}(A^* \cap B^*) = 0$. Let U, V be the components of A^*, B^* which contain A, B respectively; then, since $\cup C_i \subset U \cap V \subset \cup W_i$, we have $b_0(U \cap V) \geqq n$. But it is easy to see that $\mathrm{Fr}(U) \cap \mathrm{Fr}(V) \cap \mathrm{Fr}(U \cap V) = 0$; hence $b_0(U \cap V) \leqq r(S)$, from 7.4, and the result follows.

7.6. REMARK. The hypotheses on A and B in Theorem 9 are implied by the hypotheses in 7.3 or 7.4, and also hold whenever either $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) = 0$, or $A - B$ and $B - A$ are separated (for example, A and B both open or closed) and $\mathrm{Fr}(A \cap B) \cap \mathrm{Fr}(A \cup B) = 0$ (for example, $A \cup B = S$).

7.7. Let us now modify the definition of $r(S)$ by writing

$$r_0(S) = \sup \{ b_0(A \cap B) \mid A, B \text{ are } open, \text{ connected, and } A \cup B = S \},$$

$$r_d(S) = \sup \{ b_0(A \cap B) \mid A - B \text{ and } B - A \text{ are separated, } A, B \text{ are connected,}$$
$$\text{and } A \cup B = S \}.$$

THEOREM 10. $r_0(S) = r_d(S) = r(S)$.

In fact, it is trivial that $r_0(S) \leqq r_d(S)$, it is easy to see that $r(S) \leqq r_0(S)$ (this is shown in effect by the argument at the beginning of 4.3), and it follows from Theorem 9 (7.5) that $r_d(S) \leqq r(S)$.

7.8. *Modifications.* As one would expect, the hypotheses under which Theorem 9 was proved can be modified in various ways; for example, it would suffice that the sets A, B be sufficiently well approximable by connected sets satisfying those hypotheses. We shall here carry out one such (but less trivial) modification which will be needed later, and which has the feature that in the "interesting" case the modification is only apparent.

We need as a lemma the following modification of Theorem 8 (7.2):

LEMMA. *Let U and V be arbitrary subsets of S, and let X be a subset of $U \cap V$ such that (a) X is closed relative to $U \cap V$, (b) X is open, and (c) $\mathrm{Fr}(U) \cap \mathrm{Fr}(V) \cap \mathrm{Fr}(X) = 0$. Then there exist sets U^*, V^* such that $U^* \supset U$, $V^* \supset V$, U^* and V^* are connected relative to U and V respectively, X is closed relative to $U^* \cap V^*$, and $\mathrm{Fr}(U^*) \cap \mathrm{Fr}(V^*) \cap \mathrm{Fr}(U^* \cap V^*) = 0$. Moreover, U^* and V^* can be chosen to be open if U and V are open.*

Proof. Write $Y = (U \cap V) - X$; thus $\overline{X} \cap Y = 0$. Let Y^* be the union of all components of $Co(\overline{X})$ which meet Y; Y^* is open, connected relative to Y, contains Y, and is disjoint from X. We have only to define $U^* = U \cup Y^*$, $V^* = V \cup Y^*$; the verification of the stated properties is immediate. Incidentally, it is also clear that $U^* - V^*$ and $V^* - U^*$ will be separated if $U - V$ and $V - U$ are, and that U^* and V^* will be closed if U and V are, though in this last case the whole situation is trivial (because $X = 0$ or S).

COROLLARY. *If U and V are connected open sets, and X is a relatively open-closed subset of $U \cap V$ such that $\mathrm{Fr}(U) \cap \mathrm{Fr}(V) \cap \mathrm{Fr}(X) = 0$, then $b_0(X) \leq r(S)$; and if further $b_0(X) = r(S) < \infty$, then $X = U \cap V$.*

For, on applying 7.4 to the sets U^*, V^* defined above, we obtain $b_0(X \cup Y^*) \leq r(S)$. Since X and Y^* are disjoint open sets, this gives $b_0(X) \leq r(S)$, with finite equality only if $Y^* = 0$ and therefore $Y = 0$.

7.9. THEOREM 11. *If A and B are connected sets satisfying $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cap B) = 0$, then $b_0(A \cap B) \leq r(S)$; and if further $b_0(A \cap B) = r(S) < \infty$, then $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) = 0$.*

Suppose $b_0(A \cap B) \geq n \geq 0$; to prove the first part of the theorem it will evidently suffice to deduce that $n \leq r(S)$. We can write $A \cap B = \cup E_i$ ($1 \leq i \leq n+1$), a union of non-empty sets every two of which are separated, and can take pairwise disjoint open sets $W_i \supset E_i$. Let $\cup W_i = W$.

There exist open sets L, M, N such that $L \supset \mathrm{Fr}(A)$, $M \supset \mathrm{Fr}(B)$, $N \supset \mathrm{Fr}(A \cap B)$, and $\overline{L} \cap \overline{M} \cap \overline{N} = 0$. Again, we have $\mathrm{Cl}(A - B) \cap \mathrm{Cl}(B - A) \cap \mathrm{Cl}(A \cap B) \subset \mathrm{Fr}(A) \cap \mathrm{Fr}(B) \cap \mathrm{Fr}(A \cap B) = 0$, so that there exist open sets P, Q, R such that $P \supset \mathrm{Cl}(A - B)$, $Q \supset \mathrm{Cl}(B - A)$, $R \supset A \cap B$, and, $P \cap Q \cap R = 0$. Write $X = \{(A \cap B) \cup N\} \cap R \cap W$, $A^* = (A \cup L) \cap (P \cup X)$, and $B^* = (B \cup M) \cap (Q \cup X)$. It is easy to see that X is an open set containing $A \cap B$, from which it follows readily that A^* and B^* are open sets containing A and B respectively. Let C, D be the components of A^*, B^* which contain the respective sets A, B. Thus we have $A \cap B \subset C \cap D \subset A^* \cap B^* \subset (P \cup X) \cap (Q \cup X) = (P \cap Q) \cup X$; so, on writing $C \cap D \cap P \cap Q = H$ and $C \cap D \cap X = K$, we have $C \cap D = H \cup K$, a union of two disjoint open sets (for $H \subset P \cap Q$ and $K \subset X \subset R$), of which K contains $A \cap B$.

Now it can be verified that $\mathrm{Fr}(C) \cap \mathrm{Fr}(D) \cap \mathrm{Fr}(K) \subset \mathrm{Cl}(Co(A)) \cap \mathrm{Cl}(Co(B)) \cap \overline{L} \cap \overline{M} \cap \{\mathrm{Cl}(A \cap B) \cup \overline{N}\} = 0$. Hence the corollary to the preceding lemma shows that $b_0(K) \leq r(S)$, and if $b_0(K) = r(S) < \infty$, then $H = 0$.

Since $A \cap B \subset K \subset X$, we have $\cup E_i \subset K \subset \cup W_i$, whence $b_0(K) \geq n$. Thus $n \leq r(S)$, proving the first part of the theorem.

But if $b_0(A \cap B) = r(S) < \infty$, we may suppose that $n = b_0(A \cap B)$ in the preceding, and so have $H = 0$. But it can be verified that $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \subset C \cap D \cap P \cap Q = H$; thus $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) = 0$, completing the proof.

7.10. Similarly one can prove:

THEOREM 11a. *If A and B are connected sets such that $A-B$ and $B-A$ are separated, and $\mathrm{Fr}(A\cap B)\cap\mathrm{Fr}(A\cup B)=0$, then $b_0(A\cap B)\leq r(S)$; and if further $b_0(A\cap B)=r(S)<\infty$, then $\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\cap\mathrm{Fr}(A\cup B)=0$.*

The first part of this statement is, of course, contained in Theorem 9 (7.5); however, Theorem 11a can be established (with a suitable modification of the definition of b_0; cf. 6.6(3)) assuming only the normality, and not the complete normality, of S, whereas Theorem 9 cannot be carried over in this way as its proof uses Theorem 7a(6.5).

7.11. As an immediate consequence of Theorem 11 we have

(1) *If S is unicoherent, and A and B are intersecting connected sets, then* $\{\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\cap\mathrm{Fr}(A\cap B)=0\}\rightarrow\{\mathrm{Fr}(A)\cap\mathrm{Fr}(B)=0\}$.

Similarly, Theorem 11a gives.

(2) *If S is unicoherent, and A and B are intersecting connected sets such that $A-B$ and $B-A$ are separated, then* $\{\mathrm{Fr}(A\cap B)\cap\mathrm{Fr}(A\cup B)=0\}\rightarrow\{\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\cap\mathrm{Fr}(A\cup B)=0\}$.

It readily follows that if S is unicoherent and A and B are *open* intersecting connected sets, the four properties $\mathrm{Fr}(A\cap B)\cap\mathrm{Fr}(A\cup B)=0$, $\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\cap\mathrm{Fr}(A\cup B)=0$, $\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\cap\mathrm{Fr}(A\cap B)=0$, and $\mathrm{Fr}(A)\cap\mathrm{Fr}(B)=0$ are all equivalent.

Conversely, these implications are characteristic of unicoherence. In fact, if S is *not* unicoherent, it is easy to construct two intersecting connected open sets A, B such that $\mathrm{Fr}(A\cap B)\cap\mathrm{Fr}(A\cup B)=\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\cap\mathrm{Fr}(A\cap B)=0$, and $\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\cap\mathrm{Fr}(A\cup B)=\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\neq0$.

8. Intersections of two connected sets and their frontiers.

8.1. We conclude by generalising Theorem 5 of [U, 5.1], concerning two "nearly" connected sets whose frontiers do intersect. The result is the "expected" one:

THEOREM 12. *If $\overline{A}$ and $\overline{B}$ are connected, and $\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\neq0$, then $b_0(\overline{A}\cap\overline{B})\leq b_0\{\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\}+r(S)$.*

Proof. Suppose $b_0(A\cap B)\geq n\geq0$; it will suffice to deduce that $n\leq b_0\{\mathrm{Fr}(A)\cap\mathrm{Fr}(B)\}+r(S)$. We can write $A\cap B=C_1\cup C_2\cup\cdots\cup C_{n+1}$, where the sets C_i are closed, non-empty, and pairwise disjoint; and we can suppose the notation to be such that $C_1,\cdots,C_m$ meet $\mathrm{Fr}(A)\cap\mathrm{Fr}(B)$ and $C_{m+1},\cdots,C_{n+1}$ do not, for some m such that $1\leq m\leq n+1$. There exist open sets $W_i\supset C_i$ $(1\leq i\leq n+1)$, $H\supset\mathrm{Fr}(A)$, $K\supset\mathrm{Fr}(B)$, such that the sets $\overline{W}_i$ are pairwise disjoint and further $\overline{W}_j\cap\overline{H}\cap\overline{K}=0$ if $j>m$. Write $W=\cup W_j$ $(m<j\leq n+1)$, and let H_1 denote the union of those components of $H\cap W$ which meet $\overline{A}$. Similarly, let K_1 denote the union of the components of $K\cap W$ which meet $\overline{B}$. Write $U=\overline{A}\cup H_1$, $V=\overline{B}\cup K_1$, and $X=U\cap V\cap W$. Clearly U and V are connected; and it is easy to see that $U\cap W$ and $V\cap W$ are open, so that X is open. Again, X is closed relative to $U\cap V$, since $(U\cap V)-X=(\overline{A}\cap\overline{B})-W=C_1\cup\cdots\cup C_m=(U\cap V)\cap(W_1\cup\cdots\cup W_m)$,

which is open relative to $U \cap V$. Lastly, $\mathrm{Fr}(U) \cap \mathrm{Fr}(V) \cap \mathrm{Fr}(X) \subset \overline{H} \cap \overline{K} \cap \overline{W}$ $= 0$. Hence, by the lemma in 7.8, there exist connected sets $U^* \supset U$ and $V^* \supset V$, such that X is both open and closed relative to $U^* \cap V^*$, and $\mathrm{Fr}(U^*) \cap \mathrm{Fr}(V^*) \cap \mathrm{Fr}(U^* \cap V^*) = 0$.

By Theorem 11 (7.9), $b_0(U^* \cap V^*) \leqq r(S)$. Since the sets X and $(U^* \cap V^*)$ $-X$ are separated, and the latter is non-empty (containing C_1 at least), it follows that X has at most $r(S)$ components. But, by construction, we have $C_{m+1} \cup \cdots \cup C_{n+1} \subset X \subset W_{m+1} \cup \cdots \cup W_{n+1}$, so that X has at least $n - m + 1$ components. Thus $n - m + 1 \leqq r(S)$.

Again, for each $i \leqq m$, there exists a component F_i of $\mathrm{Fr}(A) \cap \mathrm{Fr}(B)$ which meets C_i; clearly $F_i \subset C_i$, and so $\mathrm{Fr}(A) \cap \mathrm{Fr}(B)$ has at least m components. The theorem now follows, since we have $n \leqq b_0 \{ \mathrm{Fr}(A) \cap \mathrm{Fr}(B) \} + r(S)$.

8.2. REMARKS. (1) We have shown incidentally that, under the hypotheses of Theorem 12, if $b_0(\overline{A} \cap \overline{B}) < \infty$, then all but at most $r(S)$ of the components of $\overline{A} \cap \overline{B}$ meet $\mathrm{Fr}(A) \cap \mathrm{Fr}(B)$. (Cf. [U, 5.2, Corollary 1].)

(2) Theorem 12 gives an alternative characterisation of $r(S)$. In fact, if $r(S) > n \geqq 0$, there exist open connected sets A, B such that $\mathrm{Fr}(A) \cap \mathrm{Fr}(B)$ consists of a single point and $b_0(\overline{A} \cap \overline{B}) > n$. (Cf. [U, 5.6].)

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